

**TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024**  
**SOLUTION SKETCHES TO HOMEWORK 8**

MATHIAS BRAUN AND WENHAO ZHAO

- Homework 8.1** (On Picard's little theorem). a. Employ Proposition 5.10 from the lecture notes to prove Picard's little theorem without using Picard's great theorem.  
b. Show Picard's little theorem is equivalent to the following claim. If  $f, g: \mathbf{C} \rightarrow \mathbf{C}$  are entire functions such that  $e^f + e^g = 1$ , then  $f, g$  are constant.

**Solution.** a. Let  $f: \mathbf{C} \rightarrow \mathbf{C}$  be nonconstant and holomorphic. Assume that there exist two distinct elements  $w_1, w_2 \in \mathbf{C}$  such that  $\{w_1, w_2\} \cap f(\mathbf{C}) = \emptyset$ . Then the entire assignment  $g(z) := (f(z) - w_1)(w_2 - w_1)^{-1}$  is again nonconstant and omits the values  $\{0, 1\}$ . By Proposition 5.10, we find a holomorphic function  $h: \mathbf{C} \rightarrow \mathbf{C}$  such that

$$g = \frac{1}{2} [1 + \cos(\pi(\cos \pi h))]$$

and  $h(\mathbf{C})$  contains no disc of radius larger than or equal to one. From the general version of Bloch's theorem, cf. Homework 7.1, we infer  $h$  is constant, which shows  $g$  and therefore  $f$  is constant. This is a contradiction.

b. We have seen in the proof of item a. that Picard's little theorem is equivalent to the statement that every entire function that omits the two values  $\{0, 1\}$  is constant. Assume that this statement is true. If  $e^f + e^g = 1$  for two entire functions  $f, g: \mathbf{C} \rightarrow \mathbf{C}$ , then the function  $e^g$  omits the values 0 and 1, which forces constancy of  $e^g$ . Taking the derivative we see that  $e^g g' = 0$ , so that  $g' = 0$ . Thus  $g$  is constant. Repeating the argument we conclude that  $f$  is constant.

Now assume that for all entire functions  $f, g: \mathbf{C} \rightarrow \mathbf{C}$  the equality  $e^f + e^g = 1$  implies  $f$  and  $g$  are constant. Suppose that  $h: \mathbf{C} \rightarrow \mathbf{C}$  is entire and omits the values  $\{0, 1\}$ . Then  $h = e^g$  for some entire function  $g: \mathbf{C} \rightarrow \mathbf{C}$ . Moreover, there exists an entire function  $f: \mathbf{C} \rightarrow \mathbf{C}$  such that  $1 - h = e^f$ . This entails  $e^f + e^g = 1$  and therefore  $f$  and  $g$  are constant. This implies  $h$  is constant.

- Homework 8.2** (Landau's sharpened version of Picard's little theorem\*). a. Prove there exists a function  $R: \mathbf{C} \setminus \{0, 1\} \rightarrow (0, \infty)$ <sup>1</sup> such that

$$\{f \in \mathcal{H}(\bar{B}_{R(a)}(0)) : f(0) = a, f'(0) = 1, f \text{ omits } 0 \text{ and } 1\} = \emptyset.$$

- b. Show that the statement in a. implies Picard's little theorem.

**Homework 8.3** (Approaching Picard's great theorem). Prove Picard's great theorem is equivalent to the following claim. Let  $f: B_1(0) \setminus \{0\} \rightarrow \mathbf{C}$  be any holomorphic function such that  $\{0, 1\} \cap f(B_1(0) \setminus \{0\}) = \emptyset$ . Then  $f$  or  $1/f$  is bounded in a neighborhood of 0.

**Solution.** Given Picard's great theorem, any holomorphic function  $f: B_1(0) \setminus \{0\} \rightarrow \mathbf{C}$  that omits the values 0 and 1 cannot have an essential singularity in 0. If the singularity is removable, then  $f$  is bounded in a neighborhood of 0. If  $f$  has a pole in 0, then  $1/f$  has a removable singularity in 0 and we conclude  $1/f$  is bounded in a neighborhood of 0.

To prove the converse statement, we argue by contradiction. Assume there exists a function  $f: U \setminus \{z_0\} \rightarrow \mathbf{C}$  that has an essential singularity in  $z_0$  and there exist two

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*Date:* November 18, 2024.

<sup>1</sup>**Hint.** Set  $R(a) := 3L(1/2, |a|)$ , where  $L$  is given by Schottky's theorem.

distinct values  $a, b \in \mathbb{C}$  and a neighborhood  $B_r(z_0)$  of  $z_0$  such that on  $B_r(z_0) \setminus \{z_0\}$ , the function  $f$  assumes the values  $a$  and  $b$  only finitely many times. Up to shrinking  $r$  we can assume that  $f$  omits the values  $a$  and  $b$ ; moreover, up to considering the transformation  $z \mapsto (f(rz + z_0) - a)(b - a)^{-1}$  we may and will assume  $r = 1$ ,  $z_0 = 0$ ,  $a = 0$ , and  $b = 1$ . By assumption either  $f$  or  $1/f$  is bounded in a neighborhood of 0. If  $f$  is bounded in a neighborhood of 0, it has a removable singularity in 0, which yields a contradiction. If  $g := 1/f$  is bounded, then  $g$  has a removable singularity in 0. Let  $g$  denote the nonrelabeled holomorphic extension. If  $g(0) \neq 0$ , then also  $f$  is bounded near 0 and we conclude as before. If  $g(0) = 0$  then  $\lim_{z \rightarrow 0} |f(z)| = +\infty$  and we conclude  $f$  has a pole in 0. Hence in both cases, we obtain a contradiction.

**Homework 8.4** (A stronger version of the sharpened Montel theorem). Let  $G \subset \mathbb{C}$  be a simply connected domain and for  $m \in \mathbb{N}$  define

$$\mathcal{F}_m := \{f : G \rightarrow \mathbb{C} : f \text{ holomorphic, } f(G) \cap \{0\} = \emptyset, \#\{f = 1\} \leq m\}.$$

Let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{F}_m$ . Show that either along the entire sequence  $|f_n|$  converges locally uniformly to  $\infty$  or there exists a subsequence of  $(f_n)_{n \in \mathbb{N}}$  that converges locally uniformly to a holomorphic function  $f : G \rightarrow \mathbb{C}^2$ .

**Solution.** Since  $f(G) \cap \{0\} = \emptyset$  for every  $f \in \mathcal{F}_m$  and  $G$  is simply connected, we can define the holomorphic function  $\sqrt[m+1]{f} : G \rightarrow \mathbb{C}$ . Note  $\sqrt[m+1]{f} \neq 0$  on  $G$  and moreover there exists  $s_f \in \{0, \dots, m\}$  such that for every  $z \in G$ , we have

$$\sqrt[m+1]{f(z)} \neq \exp(2\pi i s_f (m+1)^{-1}).$$

Indeed, otherwise the equation  $f(z) = 1$  has at least  $m+1$  distinct solutions — this, however, contradicts the inclusion  $f \in \mathcal{F}_m$ . Set  $z_f = \exp(2\pi i s_f) \in \partial B_1(0)$  and consider the rescaled function  $\tilde{f} := \sqrt[m+1]{f/z_f}$ . Given a sequence  $(f_n)_{n \in \mathbb{N}}$  in  $\mathcal{F}_m$  we can apply Theorem 5.12 to the sequence  $(\tilde{f}_n)_{n \in \mathbb{N}}$  and deduce either  $|\tilde{f}_n| \rightarrow \infty$  locally uniformly as  $n \rightarrow \infty$  or that along a nonrelabeled subsequence  $(\tilde{f}_n)_{n \in \mathbb{N}}$  converges locally uniformly to a holomorphic function  $\tilde{f} : G \rightarrow \mathbb{C}$ . Up to passing to a further subsequence we can also assume that in the second case  $(z_{f_n})_{n \in \mathbb{N}}$  converges to a point  $z_\infty \in \partial B_1(0)$ . This yields the claim.

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<sup>2</sup>**Hint.** Consider a suitable root  $\sqrt[k]{f}$  for  $f \in \mathcal{F}_m$ .